-Section III: Kinematics of the material point: Examples of movement

III-1 Introduction

Kinematics is the study of how a body moves, regardless of the causes that produce this displacement.

This study is based on the notions of space and time, which the observer needs to analyze the movement of objects.

Kinematics describes certain notions relating to the movements of objects, namely their trajectories, their speeds and their accelerations.These three concepts are defined or determined in a reference system.

III-2 Important notions of kinematics

To describe the movement of an object, Some important notions or concepts must be defined. It is about:

the material point : The material point is defined as being an object without spatial dimensions. This moving object can be considered as a material point when its dimensions are negligible, in front of the distances traveled by the object.

Landmark : To determine the position (location) of a material point in space, it is necessary to define a space landmark. This consists of choosing an origin O and a base $(\vec{\imath}, \vec{\jmath}, \vec{k})$, qui représente une norme ou une unité selon les trois direction de l'espace (voir section II).

Referential : It is a spatial reference , with which a temporal reference is associated (reference + clock).

Trajectory: The trajectory of a material point (M), in a given landmark, is the set of successive positions of the point M, during the movement and in the landmark.

III-3 Characteristics of a movement

The study of movement is carried out in two ways:

Vectorial : Using the vectors: position $\overrightarrow{OM}$, velocity $\vec{v}$ and acceleration $\vec{a}$.

Algebraic : By defining the equation of motion along a given trajectory.

To characterize the movement, it is necessary to clearly define the following

III-3-a The hourly equations

If a material point is in motion, its coordinates vary as a function of time. The variations of the spatial coordinates (x,y,z), over time, are called the time equations of motion. Either:

$$\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$$

III-3-b Trajectory

In a given landmark , the trajectory (C) represents the geometric location, constituted by the different positions of the material point (M) and at each instant t.

Mathematically, a trajectory is described by a relationship between the coordinates (x,y,z) of the point M, in which the time parameter t does not appear.

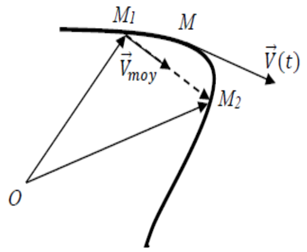
III-3-c Position vector

The position of a material point (M)at a time t is given, in a landmark, by a vector, called a position vector $\overrightarrow{OM}$. This vector relates the origin of the landmark considered to the position of the material point.

III-3-d Speed Vector or velocity

The velocity, which is a vector quantity, is defined by:

III-3-d-i Average velocity



When a material point (M) describes a trajectory C in a Referential. The point occupies the position M_1 at the instant t and the position M_2 at t'=t+□t, the average speed between t and t' is then given by : $\vec{V}(M/_R) = \frac{\overrightarrow{M_1M_2}}{t'-t} = \frac{\overrightarrow{OM_2}-\overrightarrow{OM_1}}{\Delta t}$

III-3-d-ii Instantaneous velocity

The instantaneous speed of the point (M)in the landmark (R) at an instant t is obtained by taking the limit $\Delta t \to 0$.$\vec{V}(M/_R) = \lim_{\Delta t \to 0} \frac{\overrightarrow{OM_2}-\overrightarrow{OM_1}}{\Delta t} = \left(\frac{d\overrightarrow{OM}}{dt}\right)_{/R}$

The properties of the instantaneous velocity vector are:

- Its origin which is the position of the material point at time t
- Its direction which is tangent to the trajectory at a position considered.

  Its sense, which is that of movement at the instant t.

III-3-e Acceleration Vector

Variations in speed over time define the concept of acceleration. This acceleration is defined

by:

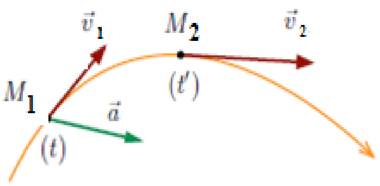
III-3-e-i Average acceleration

Let be a material point, which passes at the instant t_1 through the position M_1 at a speed $\overrightarrow{V_1}(t)$ and at the instant t_2 through the position M_2 at a speed $\overrightarrow{V_2}(t)$, on the trajectory *C*. During the interval □t=t_2-t_1, the velocity varies de $\Delta\vec{V} = \overrightarrow{V_2} - \overrightarrow{V_1}$, and the average acceleration is:

$$\vec{a}_{moy} = \frac{\overrightarrow{V_2} - \overrightarrow{V_1}}{t_2 - t_1} = \frac{\Delta\vec{V}}{\Delta t}$$

III-3-e-ii Instant acceleration

The instantaneous acceleration vector is the derivative with respect to time of the speed vector, Which is equivalent to the second derivative of the position vector: $\vec{a}(M/_R) = \lim_{\Delta t \to 0} \frac{\vec{V_2}-\vec{V_1}}{\Delta t} = \left(\frac{d\vec{V}}{dt}\right)_{/R} = \left(\frac{d^2\overrightarrow{OM}}{dt^2}\right)_{/R}$



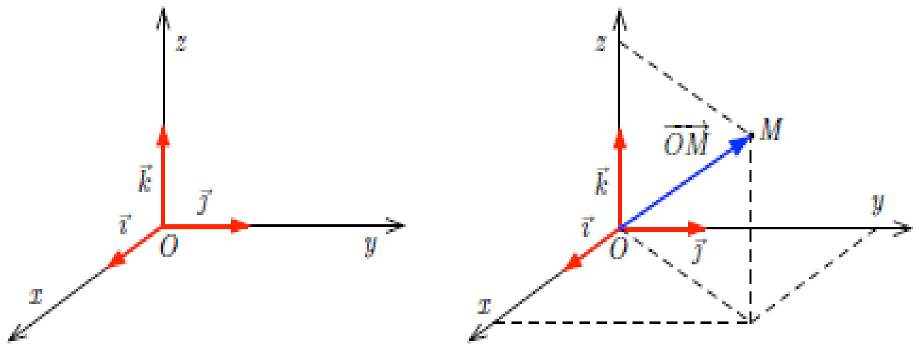
- The acceleration vector is always oriented towards the concave part of the trajectory.
- The acceleration vector describes the variations of the velocity in norm and in direction.

III-3-f Kinematic quantities in a Cartesian landmark

In this paragraph, we will define the quantities described above in the Cartesian landmark (O,x,y,z). For this we must define the notion of Galilean reference point. This is a landmark in which a free object is at rest or in uniform rectilinear motion.

III-3-f-i Position vector

In a Galilean reference frame, for example the terrestrial referential, we can attach a Cartesian landmark. (O,$\vec{i}$,$\vec{j}$,$\vec{k}$), whose basic unit vectors are fixed with respect to the referential.



(a) base cartésienne (b) vecteur position

In the Cartesian basis, the position of the point (M) is given by: $\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$.

III-3-f-ii Velocity

The expression of the velocity vector in the Cartesian basis is deduced from the relation: $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$ de sorte qu'on puisse écrire: $\vec{v} = \begin{cases} v_x = \frac{dx}{dt} \\ v_y = \frac{dy}{dt} \\ v_z = \frac{dz}{dt} \end{cases}$

the notations $\dot{x}, \dot{y}, \dot{z}$ sont souvent utilisées. The velocity is written:

$$\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$$

III-3-f-iii Acceleration vector

The acceleration vector is defined by : $\vec{a} = \frac{d\vec{V}}{dt} = \frac{dx^2}{d^2t}\vec{\imath} + \frac{dy^2}{d^2t}\vec{\jmath} + \frac{dz^2}{d^2t}\vec{k}$

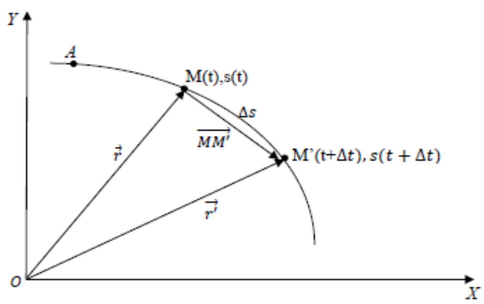
We can then write : $\vec{a} = \begin{cases} a_x = \frac{dV_x}{dt} \\ a_y = \frac{dV_y}{dt} \\ a_z = \frac{dV_z}{dt} \end{cases}$

III-3-g Kinematic quantities in the curvilinear landmark.

III-3-g-i Curvilinear abscissa

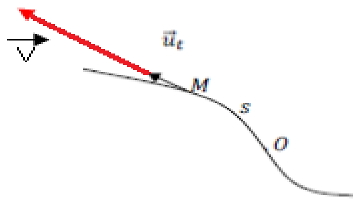
Let (M) be the position of a material point at the instant t_1 and the position M 'at the moment t_2.On call curvilinear abscissa at the moment t, noted S(t), the length of the arc of the trajectory:

S(t)= S(M) =$\widehat{MM'}(t)$.



III-3-g-ii velocity

The expression of the instantaneous speed:



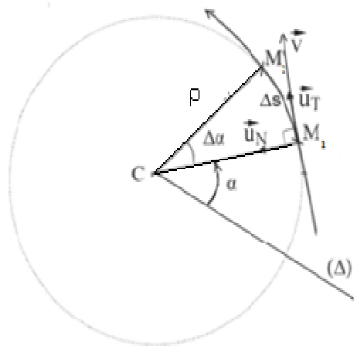
$$V(t) = \lim_{\Delta t \to 0} \frac{s(t') - s(t)}{\Delta t}$$

$$V(t) = \lim_{\Delta t \to 0} \frac{\widehat{MM'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

The module is written: $\|\vec{V}\| = \dot{S}(t)$ and the velocity is written $\vec{V} = \dot{S}(t)\overrightarrow{U_t}$.

III-3-g-iii Acceleration vector

By definition, the acceleration is given by the following equation:



$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d(v\vec{u}_t)}{dt} = \frac{dv}{dt}\vec{u}_t + v\frac{d\vec{u}_t}{dt}$$

Avec:

$$\frac{d\vec{u}_t}{dt} = \frac{d\alpha}{dt}.\frac{d\vec{u}_t}{d\alpha} = \frac{d\alpha}{ds}.\frac{ds}{dt}.\frac{d\vec{u}_t}{d\alpha}$$

$\frac{ds}{dt} = v$, $ds = \rho d\alpha$ et $\frac{d\vec{U}_t}{d\alpha} = \vec{U}_n$

$\vec{U}_n$ being the unit vector perpendicular to $\vec{U}_t$

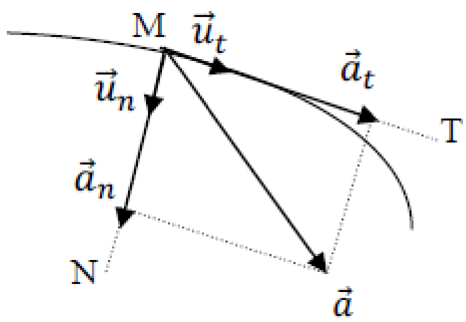
This results in the following expression of the acceleration:

$\vec{a}(t) = \frac{dv}{dt}\vec{U}_t + \frac{v^2}{\rho}\vec{U}_n$ donc $\vec{a}(t) = a_t\vec{U}_t + a_n\vec{U}_n$

Like $\vec{U}_t$ and $\vec{U}_n$ are orthogonal, the modulus of acceleration is: $a = \sqrt{{a_t}^2 + a^2{}_n}$

$\vec{a}_t$: Vector tangent to the trajectory called tangential acceleration, it indicates the variations of the modulus of the speed over time.

$\vec{a}_n$: vector normal to the trajectory called normal acceleration, it indicates the variations of the direction of the velocity vector over time.



Note :

In the case of a uniform curvilinear movement$\|\vec{V}\|$=cst⇒ $a_t = 0$ The acceleration is reduced to a single term, unlike the uniform rectilinear movement where there is no acceleration,$a_n = \frac{v^2}{\rho}$.

III-4 Frenet landmark

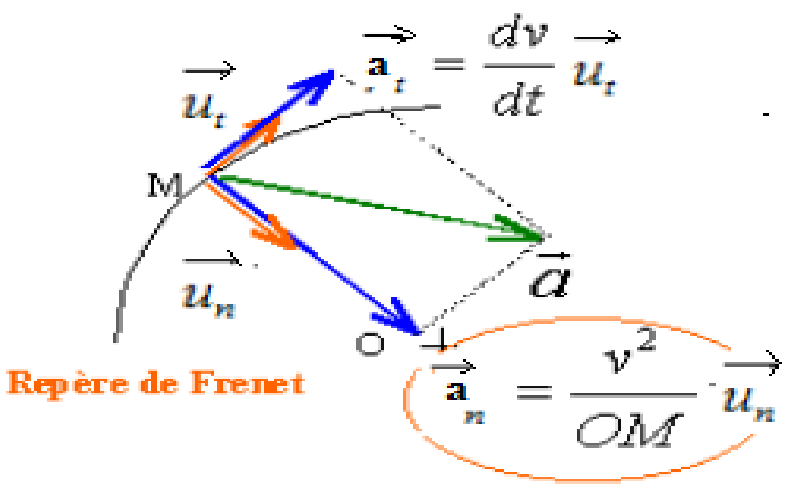
During the stady of the movement of a system, if the movement is circular, it is easier to describe it in moving reference , a particular landmark, called the frenet basis.

The Frenet ladmark is defined by :

1- A mobile origin (linkind to the point M)
2- A unit vector $\vec{U}_t$ tangent to the trajectory of the moving point and oriented in the direction of movement.
3- A unit vector $\vec{U}_n$, perpendicular to the trajectory of the moving point and oriented towards the center of the circular trajectory.

In this base, the speed and acceleration are written as follows : $\vec{v} = \|\vec{v}\|.\overrightarrow{U_t}$

$\vec{a} = \frac{d\|\vec{v}\|}{dt}.\overrightarrow{U_t} + \frac{\|\vec{v}\|^2}{\rho}.\overrightarrow{U_n}$



III-5 Kinematic quantities in some cases of movements

	Nature of the movement	Instantaneous speed	Instantaneous acceleration	Hourly equation
Rectilinear movement	Uniform rectilinear	$\vec{V} = \dot{x}\vec{\imath} = cst$	a = 0	$\frac{dx}{dt} = v$ $\int_{x_0}^{x} dx = \int_{0}^{t} v\, dt$ $x = vt + x_0$
	Uniformly Varied rectilinear	$\frac{dv}{dt} = a$ $v = at + v_0$	$\vec{a} = cst$	$x = \frac{1}{2}at^2 + v_0 t + x_0$
Circular movement	Uniform Circular	V=R. ω = cst	$a_t = 0$ $a_N = R\omega^2$	$\theta = wt + \theta_0$ $s(t) = R\dot{\theta}$
	Uniformly Varied circular	$\theta = \dot{\theta}t + \dot{\theta}_0$	$a_t = R\ddot{\theta}$ $a_N = R\dot{\theta}^2$	$\theta = \frac{1}{2}\dot{\theta}t^2 + \dot{\theta}_0 t + \theta_0$
movement sinusoïdal	Sinusoïdal rectilinear	$\vec{V} = V\vec{\imath}$	$\vec{a} = a\vec{\imath}$	$X(t) = X_0 cos(\omega t + \varphi)$ $T = \frac{2\pi}{\omega}$
	Sinusoïdal circular	$\dot{\theta} = \frac{d\theta}{dt}$	$\ddot{\theta} = \frac{d\dot{\theta}}{dt}$	$\theta(t) = \theta_0 cos(\omega t + \varphi)$ $T = \frac{2\pi}{\omega}$