SectionII: Vector calculation and coordinate systems.

II-1 Introduction

In mechanics, certain physical quantities can be completely determined by a simple number linked to a suitable unit. These quantities are scalars. Other quantities need to be oriented in one direction or located in a two-dimensional plane (2D) or in the three-dimensional space (3D).These quantities are the vectors.

As examples of scalar quantities we have time, mass, energy, temperature ... etc. As for quantities requiring a reference, we have as examples speed, acceleration, force ... etc.

II-2 The vectors

II-2-a Definition

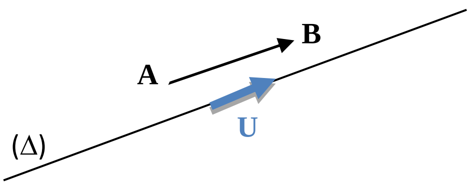
A vector is a line segment AB, having an origin A and an end B.

It is characterized by:

- Its origin or point of application A.
- Its direction (a support): It is the line which wears it. (Δ)
- Its direction: from A to B (indicated by the arrow).
- Its module which represents the length AB, which is always positive and is written$\|\overrightarrow{AB}\|$.

Each vector can be written in the form:

$\overrightarrow{AB} = \|\overrightarrow{AB}\|.\vec{U}$, $\vec{U}$ being a unit or unitary vector with a modulus equal to 1.

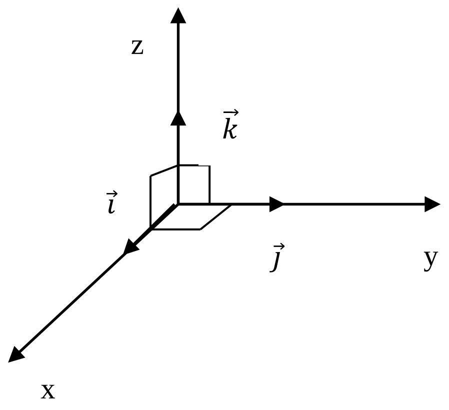


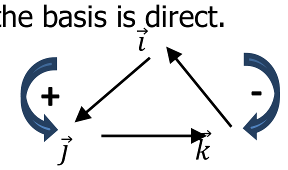
II-2-b Basis and properties of a vector

$(\vec{\imath}, \vec{\jmath}, \vec{k})$ is a direct orthonormal basis if,

- vector are two by two orthogonal.
- its elements are standardized: $\|\vec{\imath}\| =, \|\vec{J}\| = \|\vec{K}\| = 1.$
- by rotating from $\vec{\imath}$ to $\vec{\jmath}$, we progress according to $\vec{k}$. This is the rule corkscrew rule.

They say that the basis is direct.

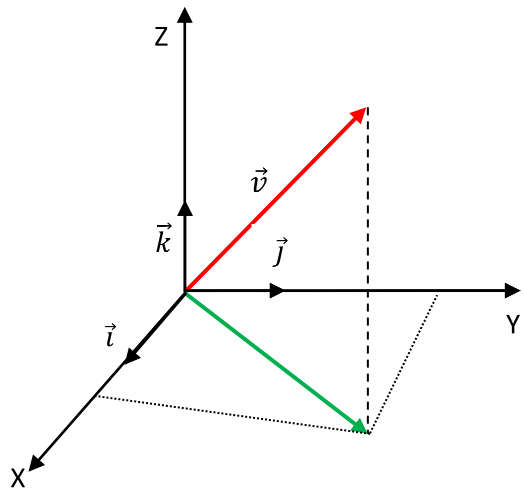




To locate a vector in three-dimensional space, it is necessary to choose a reference point of base$(\vec{\imath}, \vec{\jmath}, \vec{k})$ and of O origin.

A vector has components on the basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$ which are the projections of the vector $\vec{v}$ on the three coordinate axes X,Y and Z.

In this direct orthonormal reference, the vector $\vec{v}$ is identified by its components cartesian. $\vec{V} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ ou bien $\vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



Modulus of a vector: The standard modulus of a vector $\vec{V}$ is represented by $\|\vec{V}\|$. In the reference (O,X,Y,Z), equipped with an orthonormal system$(\vec{\imath}, \vec{\jmath}, \vec{k})$,the norm of the vector is given by:$\|\vec{V}\| = \sqrt{x^2 + y^2 + z^2}$.

In the case of a vector resulting from a sum or a difference (see the example above), the modulus of the resulting vector $\vec{A} \pm \vec{B}$ is given by:

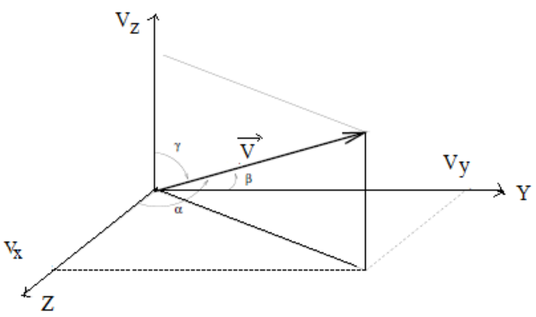
$$\|\vec{A} \pm \vec{B}\| = \sqrt{\|\vec{A}\|^2 + \|\vec{B}\|^2 \pm 2\vec{A}\vec{B}} = \sqrt{\|\vec{A}\|^2 + \|\vec{B}\|^2 \pm 2\|\vec{A}\|\|\vec{B}\| \cos\theta}$$

Directing cosines of a vector: Let α, β and γ be the angles that the vector $\vec{V}$ makes with the positive directions of the coordinate axes. If the vector V ⃗ is expressed in the following way :$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$, then the directing cosines are:

$$a = cos\alpha = \frac{\vec{V}.\vec{\imath}}{\|\vec{V}\|} = \frac{V_x}{\sqrt{V_x{}^2 + V_y{}^2 + V_z{}^2}}$$

$$b = cos\beta = \frac{\vec{V}.\vec{\jmath}}{\|\vec{V}\|} = \frac{V_y}{\sqrt{V_x{}^2 + V_y{}^2 + V_z{}^2}}$$

$$c = cos\gamma = \frac{\vec{V}.\vec{k}}{\|\vec{V}\|} = \frac{V_z}{\sqrt{V_x{}^2 + V_y{}^2 + V_z{}^2}}$$



a, b and c are the Cartesian coordinates of the unit vector $\vec{V}$ which verify the following relation: $a^2 + b^2 + c^2 = cos\alpha^2 + cos\beta^2 + cos\gamma^2 = 1$

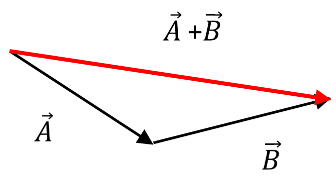
II-2-c Simple operations on vectors

II-2-c-i Sum and subtraction of two vectors

The sum of two vectors can be calculated graphically by two methods:

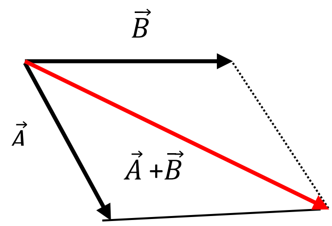
1ère method:

By the triangle. It is enough to take the end of a vector and place it at the origin of the second vector. Subsequently, reunite the origin of the first vector at the end of the second.

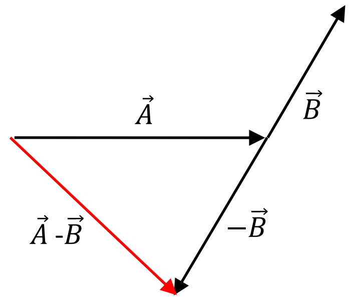


2ème method:

By the parallelogram. Place the origins of the vectors together.Complete the parallelogram.The sum vector is represented by the arrow which has as its starting point the origin of the two initial vectors and the opposite vertex of the parallelogram.



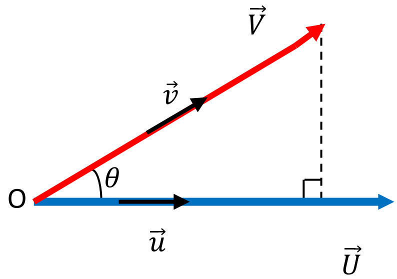
The subtraction of two vectors is done in the same way as the addition by taking the opposite of the vector B ⃗.



II-2-c-ii Scalar products of two vectors

Produit scalaire:

Let $\vec{V}$ and $\vec{U}$ be two non-zero vectors having the same origin O and making an angle θ. $\vec{v}$ et $\vec{u}$ being their respective unit vectors. the **product,** says scalar, $\vec{V}.\vec{U}$ is an algebraic number that is written:



$$\vec{U}.\vec{V} = \|\vec{U}\|.\vec{u}.\|\vec{V}\|.\vec{v} = \|\vec{U}\|.\|\vec{V}\|.cos\theta$$

By considering the components of $\vec{V}$ and $\vec{U} \Rightarrow \vec{V}: \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ et $\vec{U}: \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, the analytical expression of the Scalar product is: $\vec{V}.\vec{U} = x_1.x_2 + y_1y_2 + z_1.z_2$

The angle that the two vectors will be make : $\cos\theta = \frac{\vec{U}.\vec{V}}{\|\vec{U}\|.\|\vec{V}\|} = \frac{x_1.x_2+y_1y_2+z_1.z_2}{\|\vec{U}\|.\|\vec{V}\|}$

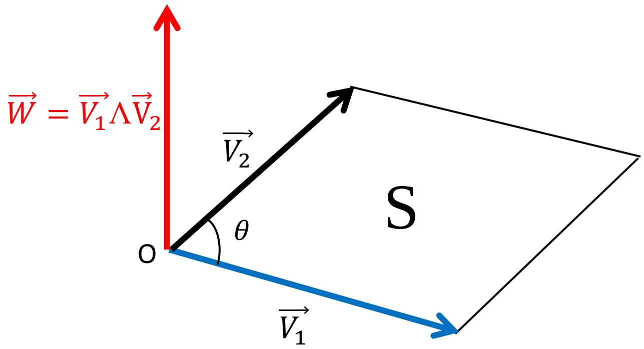
Properties of the Scalar product:

- $\vec{V}.\vec{V} = \|\vec{V}\|^2$ **:** the Scala**r** product of a vector by itself is equal to the square of its norm.
- $\vec{V}.\vec{U} = 0$ **:** if one of the two vectors is zero or the two vectors are orthogonal.
- $\alpha(\overrightarrow{V}.\overrightarrow{U}) = (\alpha.\overrightarrow{V}).\vec{U}$
- $\vec{U}.(\vec{V}+\vec{W}) = \vec{U}.\vec{V} + \vec{U}.\vec{W}$: The scalar product is distributive.

II-2-c-iii Vector product of two vectors

The Vector product of two vectors, $\vec{V_1}\Lambda\vec{V}_2$, product a vector $\vec{W}$ such as:

$$\vec{W} = \|\vec{V_1}\Lambda\vec{V}_2\| = \|\vec{V_1}\|.\|\vec{V_2}\| sin\theta$$



1-The direction of $\vec{W}$ is perpendicular to the plane formed by $\vec{V_1}$ and $\vec{V}_2$.

2- The direction of the resulting vector is direct (right hand rule-see below).

3- The norm of $\vec{W}$ is the surface of the parallelogram, denoted S in the figure.

II-2-c-iv Vector product properties - mixed product:

- The vector product is non-commutative: $\vec{V_1}\Lambda\vec{V}_2 = -\vec{V_2}\Lambda\vec{V}_1$.
- The Vector product is distributive: $\vec{V_1}\Lambda(\vec{V_2} + \vec{V_3}) = \vec{V_1}\Lambda\vec{V}_2 + \vec{V_1}\Lambda\vec{V}_3$
- If the two vectors are collinear (parallel) or one of the two null: $\vec{V_1}\Lambda\vec{V}_2 = \vec{0}$.

The vector product can also be calculated from the coordinates of the vectors:

$$\vec{V_1}: \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \vec{V}_2: \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} : \vec{V_1}\Lambda\vec{V}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \Lambda \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{vmatrix} +\vec{\imath} & -\vec{\jmath} & +\vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{pmatrix} y_1.z_2 - y_1.z_2 \\ x_2.z_1 - x_1.z_2 \\ x_1.y_2 - x_2.y_1 \end{pmatrix}$$

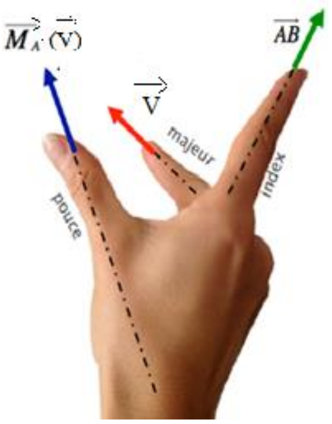
We call the mixed product of three vectors $\vec{V_1}1, \vec{V}_2$ and $\vec{V}_3$, the scalar quantity C: $C = \vec{V_3}.(\vec{V_1}\Lambda\vec{V_2})$, this quantity corresponds to the volume of the parallelepiped constructed by the three vectors (see figure above).

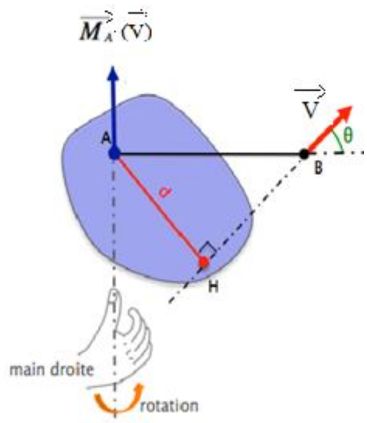
II-2-d Moment of a vector

II-2-d-i Moment of a vector with respect to a point

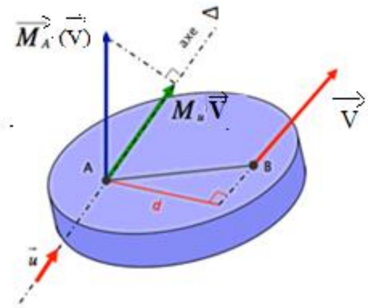
The moment of a vector $\vec{V}$ with respect to a point A is the vector $(\vec{V}) = \overrightarrow{AB} \wedge \vec{V}$. B being a point on the line of action of the vector $\vec{V}$.The moment vector is perpendicular to both $\vec{V}$ and the vector $\overrightarrow{AB}$. Its direction is given by the three-finger rule. $\vec{M}_A(\vec{V})$ is given by the expression:

$$\|\vec{M}_A(\vec{V})\| = \|\overrightarrow{AB}\|.\|\vec{V}\|.\sin(\overrightarrow{AB},\vec{V}) = V.AB.\sin\theta = V.d$$





II-2-d-i Moment of a vector with respect to an axis



The moment of a vector $\vec{V}$ with respect to an axis (Δ) of unit vector $\vec{u}$ and passing through a point A is equal to the Scalar product of the vector $\vec{u}$ by the moment in A of the vector $\vec{V}$:

$$M_\Delta(\vec{V}) = \vec{u}.\vec{M}_A(\vec{V}).$$

The moment of the vector $\vec{V}$ will be zero if the axis Δ is parallel to it.

II-2-e Derivative of a vector and derivation rules

Let be a vector$\vec{V}\ (t)$ in three-dimensional space:

$$\vec{V}(t) = V_x(t)\vec{\imath} + V_y(t)\vec{\jmath} + V_z(t)\vec{k}$$

t being a variable. The derivative of $\vec{V}\ (t)$ is given by:$\frac{d\vec{V}(t)}{dt} = \frac{d\,\vec{V}_x}{dt}\vec{\imath} + \frac{d\,\vec{V}_y}{dt}\vec{\jmath} + \frac{d\,\vec{V}_z}{dt}\vec{k}$

In the equations that follow, $\vec{V}_1(t), \vec{V}_2(t)$et $\vec{V}_3(t)$ are three vector functions depending on t. t is a scalar.

1- $\frac{d\left(\vec{V}_1(t)+\vec{V}_2(t)\right)}{dt} = \frac{d\vec{V}_1(t)}{dt} + \frac{d\vec{V}_2(t)}{dt}$

2- $\frac{d\left(\lambda\vec{V}(t)\right)}{dt} = \frac{d\lambda}{dt}\vec{V}(t) + \lambda\frac{d\vec{V}(t)}{dt}$

3- $\frac{d\left(\vec{V}_1(t).\vec{V}_2(t)\right)}{dt} = \frac{d\vec{V}_1(t)}{dt}\vec{V}_2(t) + \vec{V}_1(t)\frac{d\vec{V}_2(t)}{dt}$

4- $\frac{d(\vec{V}_1(t)\Lambda\vec{V}_2(t))}{dt} = \frac{d\vec{V}_1(t)}{dt}\Lambda\vec{V}_2(t) + \vec{V}_1(t)\Lambda\frac{d\vec{V}_2(t)}{dt}$